

# Complete $(p, q)$ -elliptic integrals with application to a family of means <sup>\*</sup>

Toshiki Kamiya and Shingo Takeuchi<sup>†</sup>  
Department of Mathematical Sciences  
Shibaura Institute of Technology <sup>‡</sup>

## Abstract

The complete elliptic integrals are generalized by using the generalized trigonometric functions with two parameters. It is shown that a particular relation holds for the generalized integrals. Moreover, as an application of the integrals, an alternative proof of a result for a family of means by Bhatia and Li, which involves the logarithmic mean and the arithmetic-geometric mean, is given.

**Keywords:** Complete elliptic integrals, Generalized trigonometric functions, Arithmetic-Geometric mean, Logarithmic mean, Gaussian hypergeometric functions,  $p$ -Laplacian.

## 1 Introduction

In this paper, we deal with a *complete  $(p, q)$ -elliptic integral of the first kind*

$$\mathcal{K}_{p,q}(k) := \int_0^{\frac{\pi p,q}{2}} \frac{d\theta}{(1 - k^q \sin_{p,q}^q \theta)^{1-\frac{1}{p}}} = \int_0^1 \frac{dt}{(1 - t^q)^{\frac{1}{p}} (1 - k^q t^q)^{1-\frac{1}{p}}},$$

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<sup>\*</sup>The work of S. Takeuchi was supported by JSPS KAKENHI Grant Number 24540218.

<sup>†</sup>Corresponding author

<sup>‡</sup>307 Fukasaku, Minuma-ku, Saitama-shi, Saitama 337-8570, Japan.

*E-mail address:* shingo@shibaura-it.ac.jp (S. Takeuchi)

*2010 Mathematics Subject Classification.* 34L10, 33E05, 33C75

where  $\sin_{p,q} \theta$  is the generalized  $(p, q)$ -trigonometric function and  $\pi_{p,q}$  denotes the half-period of  $\sin_{p,q} \theta$ . The function  $\sin_{p,q} \theta$  and the number  $\pi_{p,q}$  play important roles to express the solutions  $(\lambda, u)$  of inhomogeneous eigenvalue problem of  $p$ -Laplacian  $-(|u'|^{p-2}u')' = \lambda|u|^{q-2}u$  with a boundary condition. See Section 2 for the definition of  $\sin_{p,q} \theta$  and  $\pi_{p,q}$ ; also [4, 5, 7, 8] for details. For  $p = q = 2$ , it is easy to see that  $\sin_{p,q} \theta$ ,  $\pi_{p,q}$  and  $\mathcal{K}_{p,q}(k)$  are identical to the classical  $\sin \theta$ ,  $\pi$  and  $\mathcal{K}(k)$ , respectively, where  $\mathcal{K}(k)$  is the complete elliptic integral of the first kind

$$\mathcal{K}(k) := \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}}.$$

Moreover,  $\mathcal{K}_{p,q}(k)$  for  $p = q$  has been already studied in [9].

In this paper we will apply the complete  $(p, q)$ -elliptic integral  $\mathcal{K}_{p,q}(k)$  to study a family of means defined by Bhatia and Li [2] and to give an alternative proof of their theorem.

For a while, we will describe a part of the study in [2]. Let  $a$  and  $b$  be positive numbers. The *logarithmic mean*  $L(a, b)$  of  $a$  and  $b$  is defined by

$$L(a, b) := \begin{cases} \frac{a - b}{\log a - \log b} & (a \neq b), \\ a & (a = b). \end{cases}$$

The *arithmetic-geometric mean*  $AG(a, b)$  of  $a$  and  $b$  is defined as follows: Let us consider the sequences  $\{a_n\}$  and  $\{b_n\}$  satisfying

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}, \quad n = 0, 1, 2, \dots$$

with  $a_0 = a$  and  $b_0 = b$ . The sequences  $\{a_n\}$  and  $\{b_n\}$  converge to a common limit, and

$$AG(a, b) := \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n.$$

It is known that  $L(a, b)$  and  $AG(a, b)$  have integral expressions as

$$\frac{1}{L(a, b)} = \int_0^\infty \frac{dt}{(t + a)(t + b)},$$

$$\frac{1}{AG(a, b)} = \frac{2}{\pi} \int_0^\infty \frac{dt}{\sqrt{(t^2 + a^2)(t^2 + b^2)}}.$$

Indeed, the first one follows from direct calculation of the right-hand side and the second one is a celebrated result of Gauss (e.g., [1, Theorem 3.2.3] or [3, Theorem 1.1] with setting  $b \tan \theta = t$ ).

Motivated by these expressions, Bhatia and Li introduced an interpolating family of means  $M_p(a, b)$  by

$$\frac{1}{M_p(a, b)} := c_p \int_0^\infty \frac{dt}{((t^p + a^p)(t^p + b^p))^{\frac{1}{p}}}, \quad p \in (0, \infty),$$

where  $c_p$  is defined to satisfy  $M_p(a, a) = a$ , hence,

$$\frac{1}{c_p} := \int_0^\infty \frac{dt}{(1 + t^p)^{\frac{2}{p}}}.$$

Moreover,  $M_0$  is defined by taking limit:

$$M_0(a, b) = \lim_{p \rightarrow +0} M_p(a, b) = \sqrt{ab}.$$

Clearly,  $M_1(a, b) = L(a, b)$  and  $M_2(a, b) = AG(a, b)$ , thus  $M_p(a, b)$  is a generalization of  $L(a, b)$  and  $AG(a, b)$ . It is easily seen that  $M_p(a, b)$  is a *binary symmetric mean* of positive numbers  $a$  and  $b$ , that is

- (i)  $\min\{a, b\} \leq M_p(a, b) \leq \max\{a, b\}$
- (ii)  $M_p(a, b) = M_p(b, a)$
- (iii)  $M_p(\alpha a, \alpha b) = \alpha M_p(a, b)$  for all  $\alpha > 0$
- (iv)  $M_p(a, b)$  is non-decreasing in  $a$  and  $b$ .

They studied relation between  $M_p(a, b)$  and  $K_p(a, b)$ , the *power difference mean* of  $a$  and  $b$ . This is defined for any  $p \in \mathbb{R}$  and  $a, b > 0$  by

$$K_p(a, b) := \begin{cases} \frac{p-1}{p} \frac{a^p - b^p}{a^{p-1} - b^{p-1}} & (a \neq b), \\ a & (a = b), \end{cases}$$

where it is understood that

$$\begin{aligned} K_0(a, b) &:= \lim_{p \rightarrow 0} K_p(a, b) = \frac{ab}{L(a, b)}, \\ K_1(a, b) &:= \lim_{p \rightarrow 1} K_p(a, b) = L(a, b). \end{aligned}$$

For more details of  $K_p(a, b)$ , see [2, 6] and the references given there.

These two means are related in the following sense. It is easy to check that  $1/M_p(a, b)$  can be written as ( $t^p + a^p = a^p s^{-1}$ )

$$\frac{1}{M_p(a, b)} = \frac{\int_0^1 \frac{s^{\frac{1}{p}-1}(1-s)^{\frac{1}{p}-1}}{(a^p(1-s) + b^p s)^{\frac{1}{p}}} ds}{\int_0^1 s^{\frac{1}{p}-1}(1-s)^{\frac{1}{p}-1} ds}$$

and  $1/K_p(a, b)$  also admits the following integral expression:

$$\frac{1}{K_p(a, b)} = \int_0^1 \frac{ds}{(a^p(1-s) + b^p s)^{\frac{1}{p}}}. \quad (1.1)$$

Thus  $1/M_p(a, b)$  and  $1/K_p(a, b)$  are the weighted mean with the beta distribution and with the continuous uniform distribution of the function  $(a^p(1-s) + b^p s)^{-1/p}$ , respectively.

They concluded the following theorem with easy but technical calculation. But, it is hard to say that these calculations are natural.

**Theorem 1.1** ([2]). *Given  $a, b > 0$  and  $a \neq b$ , we have*

- (i)  $M_p(a, b) > K_p(a, b)$  if  $0 \leq p < 1$
- (ii)  $M_1(a, b) = K_1(a, b)$
- (iii)  $M_p(a, b) < K_p(a, b)$  if  $p > 1$ .

In this paper, we will give an alternative proof of Theorem 1.1. Using the complete  $(p, q)$ -elliptic integral, we can easily give a hypergeometric representation (1.2) in Theorem 1.2 below for  $1/M_p(a, b)$ . Applying a formula of hypergeometric function to (1.2) and (1.4), we have (1.3) and (1.5). We emphasize that Theorem 1.1 of Bhatia and Li follows immediately from Theorem 1.2 with comparing only the third parameters of (1.3) and (1.5).

**Theorem 1.2.** *Let  $p \in (0, \infty)$ ,  $p \neq 1$  and  $x \in (0, 1]$ . Then*

$$\begin{aligned} \frac{1}{M_p(1, x)} &= \frac{2}{\pi_{p^*, p}} \mathcal{K}_{p^*, p}((1 - x^p)^{\frac{1}{p}}) \\ &= F\left(\frac{1}{p}, \frac{1}{p}; \frac{2}{p}; 1 - x^p\right), \end{aligned} \quad (1.2)$$

$$= \left(\frac{1 + x^p}{2}\right)^{-\frac{1}{p}} F\left(\frac{1}{2p}, \frac{1}{2p} + \frac{1}{2}; \frac{1}{p} + \frac{1}{2}; \left(\frac{1 - x^p}{1 + x^p}\right)^2\right), \quad (1.3)$$

$$\frac{1}{K_p(1, x)} = F\left(1, \frac{1}{p}; 2; 1 - x^p\right), \quad (1.4)$$

$$= \left(\frac{1 + x^p}{2}\right)^{-\frac{1}{p}} F\left(\frac{1}{2p}, \frac{1}{2p} + \frac{1}{2}; \frac{3}{2}; \left(\frac{1 - x^p}{1 + x^p}\right)^2\right). \quad (1.5)$$

Therefore, Theorem 1.1 immediately follows.

Moreover, we define a complete  $(p, q)$ -elliptic integral of the second kind

$$\mathcal{E}_{p, q}(k) := \int_0^{\frac{\pi_{p, q}}{2}} (1 - k^q \sin_{p, q}^q \theta)^{\frac{1}{p}} d\theta = \int_0^1 \left(\frac{1 - k^q t^q}{1 - t^q}\right)^{\frac{1}{p}} dt.$$

It is clear that  $\mathcal{E}_{2, 2}(k)$  is identical to the complete elliptic integral of the second kind

$$\mathcal{E}(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \theta} d\theta = \int_0^1 \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} dt.$$

Then, we can show the following relation for  $p \neq q$ .

**Theorem 1.3.** *Let  $p, q \in (1, \infty)$  and  $k \in [0, 1)$ . Then*

$$p\mathcal{E}_{p, q}(k^{\frac{1}{q}})\mathcal{K}_{q, p}(k^{\frac{1}{p}}) - q\mathcal{K}_{p, q}(k^{\frac{1}{q}})\mathcal{E}_{q, p}(k^{\frac{1}{p}}) = \frac{(p - q)\pi_{p, q}\pi_{q, p}}{4}. \quad (1.6)$$

This paper is organized as follows. In Section 2 we prepare properties of the complete  $(p, q)$ -elliptic integrals and show Theorem 1.3. Section 3 is devoted to give a proof of Theorem 1.2 and an alternative proof of Theorem 1.1.

Throughout this paper, we write  $\mathbb{P} := (0, 1) \cup (1, \infty)$ .

## 2 Complete $(p, q)$ -elliptic integrals

Let  $p$  and  $q$  be real numbers satisfying  $p^* := p/(p-1) > 0$  and  $q > 0$  (note that  $p$  is allowed to be negative). The  $(p, q)$ -trigonometric function  $\sin_{p,q} x$  is the inverse function of

$$\sin_{p,q}^{-1} x := \int_0^x \frac{dt}{(1-t^q)^{\frac{1}{p}}}, \quad x \in [0, 1].$$

Clearly,  $\sin_{p,q} x$  is increasing function from  $[0, \pi_{p,q}/2]$  onto  $[0, 1]$ , where

$$\pi_{p,q} := 2 \sin_{p,q}^{-1} 1 = 2 \int_0^1 \frac{dt}{(1-t^q)^{\frac{1}{p}}} = \frac{2}{q} B\left(\frac{1}{p^*}, \frac{1}{q}\right).$$

For  $x \in [0, \pi_{p,q}/2)$ , we also define

$$\cos_{p,q} x := (1 - \sin_{p,q}^q x)^{\frac{1}{q}}, \quad \tan_{p,q} x := \frac{\sin_{p,q} x}{\cos_{p,q} x}.$$

These functions satisfy, for  $x \in (0, \pi_{p,q}/2)$ ,

$$\begin{aligned} \cos_{p,q}^q x + \sin_{p,q}^q x &= 1, \\ (\sin_{p,q} x)' &= \cos_{p,q}^{\frac{q}{p}} x, \\ (\cos_{p,q} x)' &= -\sin_{p,q}^{q-1} x \cos_{p,q}^{1-\frac{q}{p^*}} x, \\ (\cos_{p,q}^{\frac{q}{p^*}} x)' &= -\frac{q}{p^*} \sin_{p,q}^{q-1} x, \\ (\tan_{p,q} x)' &= \cos_{p,q}^{-1-\frac{q}{p^*}} x. \end{aligned}$$

Now, for any  $k \in [0, 1)$  we define the *complete  $(p, q)$ -elliptic integral of the first kind* and *of the second kind* as follows.

$$\begin{aligned} \mathcal{K}_{p,q}(k) &:= \int_0^{\frac{\pi_{p,q}}{2}} \frac{d\theta}{(1 - k^q \sin_{p,q}^q \theta)^{\frac{1}{p^*}}} = \int_0^1 \frac{dt}{(1-t^q)^{\frac{1}{p}} (1 - k^q t^q)^{\frac{1}{p^*}}}, \\ \mathcal{E}_{p,q}(k) &:= \int_0^{\frac{\pi_{p,q}}{2}} (1 - k^q \sin_{p,q}^q \theta)^{\frac{1}{p}} d\theta = \int_0^1 \left( \frac{1 - k^q t^q}{1 - t^q} \right)^{\frac{1}{p}} dt. \end{aligned}$$

It is easy to see that  $\mathcal{K}_{p,q}(k)$  is increasing on  $[0, 1)$  and

$$\mathcal{K}_{p,q}(0) = \frac{\pi_{p,q}}{2}, \quad \lim_{k \rightarrow 1-0} \mathcal{K}_{p,q}(k) = \infty,$$

and  $\mathcal{E}_{p,q}(k)$  is decreasing on  $[0, 1)$  and

$$\mathcal{E}_{p,q}(0) = \frac{\pi_{p,q}}{2}, \quad \lim_{k \rightarrow 1-0} \mathcal{E}_{p,q}(k) = 1,$$

The functions  $\mathcal{K}_{p,q}(k)$  and  $\mathcal{E}_{p,q}(k)$  satisfy a system of differential equations.

**Proposition 2.1.**

$$\frac{d\mathcal{E}_{p,q}}{dk} = \frac{q(\mathcal{E}_{p,q} - \mathcal{K}_{p,q})}{pk}, \quad \frac{d\mathcal{K}_{p,q}}{dk} = \frac{\mathcal{E}_{p,q} - (1 - k^q)\mathcal{K}_{p,q}}{k(1 - k^q)}.$$

*Proof.* Differentiating  $\mathcal{E}_{p,q}(k)$  we have

$$\begin{aligned} \frac{d\mathcal{E}_{p,q}}{dk} &= \int_0^{\frac{\pi_{p,q}}{2}} \frac{d}{dk} (1 - k^q \sin_{p,q}^q \theta)^{\frac{1}{p}} d\theta \\ &= \frac{q}{p} \int_0^{\frac{\pi_{p,q}}{2}} \frac{-k^{q-1} \sin_{p,q}^q \theta}{(1 - k^q \sin_{p,q}^q \theta)^{1-\frac{1}{p}}} d\theta \\ &= \frac{q}{pk} \left( \int_0^{\frac{\pi_{p,q}}{2}} \frac{1 - k^q \sin_{p,q}^q \theta}{(1 - k^q \sin_{p,q}^q \theta)^{1-\frac{1}{p}}} d\theta - \int_0^{\frac{\pi_{p,q}}{2}} \frac{d\theta}{(1 - k^q \sin_{p,q}^q \theta)^{1-\frac{1}{p}}} \right) \\ &= \frac{q}{pk} (\mathcal{E}_{p,q} - \mathcal{K}_{p,q}). \end{aligned}$$

Next, for  $\mathcal{K}_{p,q}(k)$

$$\frac{d\mathcal{K}_{p,q}}{dk} = \frac{q}{p^*} \int_0^{\frac{\pi_{p,q}}{2}} \frac{k^{q-1} \sin_{p,q}^q \theta}{(1 - k^q \sin_{p,q}^q \theta)^{2-\frac{1}{p}}} d\theta. \quad (2.1)$$

Here we see that

$$\begin{aligned} &\frac{d}{d\theta} \left( \frac{-\cos_{p,q}^{\frac{q}{p^*}} \theta}{(1 - k^q \sin_{p,q}^q \theta)^{1-\frac{1}{p}}} \right) \\ &= \frac{\frac{q}{p^*} \sin_{p,q}^{q-1} \theta (1 - k^q \sin_{p,q}^q \theta) - \frac{q}{p^*} k^q \sin_{p,q}^{q-1} \theta \cos_{p,q}^q \theta}{(1 - k^q \sin_{p,q}^q \theta)^{2-\frac{1}{p}}} \\ &= \frac{q(1 - k^q) \sin_{p,q}^{q-1} \theta}{p^*(1 - k^q \sin_{p,q}^q \theta)^{2-\frac{1}{p}}}, \end{aligned}$$

so that we use integration by parts as

$$\begin{aligned}
\frac{d\mathcal{K}_{p,q}}{dk} &= \int_0^{\frac{\pi_{p,q}}{2}} \frac{k^{q-1}}{1-k^q} \frac{d}{d\theta} \left( \frac{-\cos_{p,q}^{\frac{q}{p}} \theta}{(1-k^q \sin_{p,q}^q \theta)^{1-\frac{1}{p}}} \right) \sin_{p,q} \theta d\theta \\
&= \frac{k^{q-1}}{1-k^q} \left[ \frac{-\cos_{p,q}^{\frac{q}{p}} \theta \sin_{p,q} \theta}{(1-k^q \sin_{p,q}^q \theta)^{1-\frac{1}{p}}} \right]_0^{\frac{\pi_{p,q}}{2}} + \frac{k^{q-1}}{1-k^q} \int_0^{\frac{\pi_{p,q}}{2}} \frac{\cos_{p,q}^q \theta}{(1-k^q \sin_{p,q}^q \theta)^{1-\frac{1}{p}}} d\theta \\
&= \frac{k^{q-1}}{1-k^q} \int_0^{\frac{\pi_{p,q}}{2}} \frac{1}{k^q} \cdot \frac{1-k^q \sin_{p,q}^q \theta - (1-k^q)}{(1-k^q \sin_{p,q}^q \theta)^{1-\frac{1}{p}}} d\theta \\
&= \frac{1}{k(1-k^q)} (\mathcal{E}_{p,q} - (1-k^q) \mathcal{K}_{p,q}).
\end{aligned}$$

This completes the proof.  $\square$

Proposition 2.1 now yields Theorem 1.3.

*Proof of Theorem 1.3.* We will differentiate the left-hand side of (1.6) and apply Proposition 2.1. A direct computation shows that

$$\begin{aligned}
&\frac{d}{dk} (p\mathcal{E}_{p,q}(k^{\frac{1}{q}}) \mathcal{K}_{q,p}(k^{\frac{1}{p}}) - q\mathcal{K}_{p,q}(k^{\frac{1}{q}}) \mathcal{E}_{q,p}(k^{\frac{1}{p}})) \\
&= p \cdot \frac{1}{pk} (\mathcal{E}_{p,q}(k^{\frac{1}{q}}) - \mathcal{K}_{p,q}(k^{\frac{1}{q}})) \cdot \mathcal{K}_{q,p}(k^{\frac{1}{p}}) \\
&\quad + p\mathcal{E}_{p,q}(k^{\frac{1}{q}}) \cdot \frac{1}{pk(1-k)} (\mathcal{E}_{q,p}(k^{\frac{1}{p}}) - (1-k)\mathcal{K}_{q,p}(k^{\frac{1}{p}})) \\
&\quad - q \cdot \frac{1}{qk(1-k)} (\mathcal{E}_{p,q}(k^{\frac{1}{q}}) - (1-k)\mathcal{K}_{p,q}(k^{\frac{1}{q}})) \cdot \mathcal{E}_{q,p}(k^{\frac{1}{p}}) \\
&\quad - q\mathcal{K}_{p,q}(k^{\frac{1}{q}}) \cdot \frac{1}{qk} (\mathcal{E}_{q,p}(k^{\frac{1}{p}}) - \mathcal{K}_{q,p}(k^{\frac{1}{p}})) \\
&= 0.
\end{aligned}$$

Therefore the left-hand side of (1.6) is a constant  $C$ . Letting  $k = 0$ , we obtain

$$C = p \frac{\pi_{p,q}}{2} \frac{\pi_{q,p}}{2} - q \frac{\pi_{p,q}}{2} \frac{\pi_{q,p}}{2} = \frac{(p-q)\pi_{p,q}\pi_{q,p}}{4},$$

and the proof is complete.  $\square$



For a real number  $a$  and a natural number  $n$ , we define

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = (a+n-1)(a+n-2)\cdots(a+1)a.$$

We adopt the convention that  $(a)_0 := 1$ . For  $|x| < 1$  the series

$$F(a, b; c; x) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}$$

is called a *Gaussian hypergeometric series*. See [1] for more details.

**Lemma 2.2.** *For  $n = 0, 1, 2, \dots$*

$$\int_0^{\frac{\pi_{p,q}}{2}} \sin_{p,q}^{qn} \theta \, d\theta = \frac{\pi_{p,q}}{2} \frac{(\frac{1}{q})_n}{(\frac{1}{p^*} + \frac{1}{q})_n}.$$

*Proof.* Letting  $\sin_{p,q}^q \theta = t$ , we have

$$\int_0^{\frac{\pi_{p,q}}{2}} \sin_{p,q}^{qn} \theta \, d\theta = \frac{1}{q} \int_0^1 t^{n+\frac{1}{q}-1} (1-t)^{-\frac{1}{p}} dt = \frac{1}{q} B\left(n + \frac{1}{q}, \frac{1}{p^*}\right).$$

Moreover,

$$\begin{aligned} \frac{1}{q} B\left(n + \frac{1}{q}, \frac{1}{p^*}\right) &= \frac{1}{q} B\left(\frac{1}{q}, \frac{1}{p^*}\right) \frac{B\left(n + \frac{1}{q}, \frac{1}{p^*}\right)}{B\left(\frac{1}{q}, \frac{1}{p^*}\right)} \\ &= \frac{\pi_{p,q}}{2} \frac{\Gamma(n + \frac{1}{q}) \Gamma(\frac{1}{q} + \frac{1}{p^*})}{\Gamma(\frac{1}{q}) \Gamma(n + \frac{1}{q} + \frac{1}{p^*})} \\ &= \frac{\pi_{p,q}}{2} \frac{(\frac{1}{q})_n}{(\frac{1}{p^*} + \frac{1}{q})_n}, \end{aligned}$$

and the lemma follows. □

**Proposition 2.3.**

$$\begin{aligned} \mathcal{K}_{p,q}(k) &= \frac{\pi_{p,q}}{2} F\left(\frac{1}{p^*}, \frac{1}{q}; \frac{1}{p^*} + \frac{1}{q}; k^q\right), \\ \mathcal{E}_{p,q}(k) &= \frac{\pi_{p,q}}{2} F\left(-\frac{1}{p}, \frac{1}{q}; \frac{1}{p^*} + \frac{1}{q}; k^q\right). \end{aligned}$$

*Proof.* Binomial series expansion gives

$$\mathcal{K}_{p,q}(k) = \int_0^{\frac{\pi_{p,q}}{2}} (1 - k^q \sin_{p,q}^q \theta)^{-\frac{1}{p^*}} d\theta = \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{p^*}}{n} k^{qn} \int_0^{\frac{\pi_{p,q}}{2}} \sin_{p,q}^{qn} \theta d\theta.$$

Here, using Lemma 2.2 and the fact

$$(-1)^n \binom{-\frac{1}{p^*}}{n} = \frac{(\frac{1}{p^*})_n}{n!},$$

we see that

$$\mathcal{K}_{p,q}(k) = \frac{\pi_{p,q}}{2} \sum_{n=0}^{\infty} \frac{(\frac{1}{p^*})_n (\frac{1}{q})_n}{(\frac{1}{p^*} + \frac{1}{q})_n} \frac{k^{qn}}{n!} = \frac{\pi_{p,q}}{2} F\left(\frac{1}{p^*}, \frac{1}{q}; \frac{1}{p^*} + \frac{1}{q}; k^q\right).$$

The proof of  $\mathcal{E}_{p,q}(k)$  is similar, so that we omit it.  $\square$

### 3 Proof of Theorem 1.2

By properties (ii) and (iii) of binary symmetric mean in the introduction, we may assume that  $a \geq b > 0$  and it is enough to consider  $M_p(1, x)$  for any  $x \in (0, 1]$  instead of  $M_p(a, b)$ .

The following is a fundamental quadratic transformation of hypergeometric functions. For the proof, see for instance [1, Theorem 3.1.3].

**Lemma 3.1.** *For all  $x$  where the series converge*

$$F(a, b; 2a; x) = \left(1 - \frac{x}{2}\right)^{-b} F\left(\frac{b}{2}, \frac{b+1}{2}; a + \frac{1}{2}; \left(\frac{x}{2-x}\right)^2\right).$$

Now we are in a position to prove Theorem 1.2. Let  $p \in \mathbb{P}$ . Setting  $t = x \tan_{p^*,p} \theta$  in the right-hand side of

$$\frac{1}{M_p(1, x)} = c_p \int_0^{\infty} \frac{dt}{((t^p + 1)(t^p + x^p))^{\frac{1}{p}}},$$

we have

$$\begin{aligned}
\frac{1}{M_p(1, x)} &= c_p \int_0^{\frac{\pi_{p^*, p}}{2}} \frac{x \cos_{p^*, p}^{-2} \theta d\theta}{(x^p \tan_{p^*, p}^p \theta + 1)^{\frac{1}{p}} (x^p \tan_{p^*, p}^p \theta + x^p)^{\frac{1}{p}}} \\
&= c_p \int_0^{\frac{\pi_{p^*, p}}{2}} \frac{d\theta}{(\cos_{p^*, p}^p \theta + x^p \sin_{p^*, p}^p \theta)^{\frac{1}{p}}} \\
&= c_p \int_0^{\frac{\pi_{p^*, p}}{2}} \frac{d\theta}{(1 - (1 - x^p) \sin_{p^*, p}^p \theta)^{\frac{1}{p}}} \\
&= c_p \mathcal{K}_{p^*, p}((1 - x^p)^{\frac{1}{p}}),
\end{aligned}$$

where

$$\frac{1}{c_p} = \int_0^\infty \frac{dt}{(1 + t^p)^{\frac{2}{p}}} = \frac{\pi_{p^*, p}}{2}.$$

Thus, Proposition 2.3 yields (1.2), i.e.,

$$\frac{1}{M_p(1, x)} = F\left(\frac{1}{p}, \frac{1}{p}; \frac{2}{p}; 1 - x^p\right).$$

Applying Lemma 3.1 with  $a = b = 1/p$  and  $x$  replaced by  $1 - x^p$  to (1.2), we have (1.3).

Next, recall that  $1/K_p(1, x)$  can be written as (1.1). As in the proof of Proposition 2.3, we obtain

$$\begin{aligned}
\frac{1}{K_p(1, x)} &= \int_0^1 (1 - (1 - x^p)s)^{-\frac{1}{p}} ds \\
&= \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{p}}{n} (1 - x^p)^n \int_0^1 s^n ds \\
&= \sum_{n=0}^{\infty} \frac{(\frac{1}{p})_n}{(n+1)!} (1 - x^p)^n \\
&= F\left(1, \frac{1}{p}; 2; 1 - x^p\right),
\end{aligned}$$

which implies (1.4). Applying Lemma 3.1 with  $a = 1$ ,  $b = 1/p$  and  $x$  replaced by  $1 - x^p$  to the last series, we have (1.5). Therefore, we accomplished the proof of Theorem 1.2.

Theorem 1.1 immediately follows from Theorem 1.2. Indeed, we assume  $p \in \mathbb{P}$ . Comparing the third parameters of (1.3) and (1.5), we can see that

$$\frac{1}{M_p(1, x)} \geq \frac{1}{K_p(1, x)} \Leftrightarrow \frac{1}{p} + \frac{1}{2} \leq \frac{3}{2},$$

hence

$$M_p(1, x) \geq K_p(1, x) \Leftrightarrow p \leq 1.$$

We leave it to the reader to verify that  $M_0(a, b) > K_0(a, b)$ .

*Remark 3.2.* Motivated by the expression in [2]:

$$\frac{1}{M_p(a, b)} = (\max\{a, b\})^{-1} \sum_{k=0}^{\infty} \prod_{i=0}^{k-1} \frac{(\frac{1}{p} + i)^2}{\frac{2}{p} + i} \frac{1}{k!} \left[ 1 - \left( \frac{\min\{a, b\}}{\max\{a, b\}} \right)^p \right]^k, \quad (3.1)$$

Nakamura [6, Remark 3.9] also indicates that  $1/M_p(1, x)$  is nothing but (1.2). On the other hand, our proof gives (1.2) without deducing (3.1).

*Remark 3.3.* Each mean of  $L(a, b)$ ,  $AG(a, b)$  and  $K_p(a, b)$  has the other characterization than the integral expression. It is of interest to characterize Bhatia-Li's mean  $M_p(a, b)$  with no use of integral, but we have not been able to do this.

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